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# MODIFIED MORREY SPACES FOR THE DUNKL OPERATOR ON THE REAL LINE AND THE FRACTIONAL MAXIMAL FUNCTION

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**Abstract.** On the real line, the Dunkl operators $D_\nu$ are differential-difference operators associated with the reflection group $Z_2$ on $R$. In this paper, we study the fractional maximal operator associated with the Dunkl operator $M_{\alpha,\nu}$ in modified $D_\nu$-Morrey spaces $\widetilde{L}_{p,\lambda}(R, dm_\nu)$ within the framework of $R$. We present sufficient conditions for the boundedness of the operator $M_{\alpha,\nu}$ on modified $D_\nu$ - Morrey spaces $\widetilde{L}_{p,\lambda}(R, dm_\nu)$.

**Keywords:** Fractional maximal operator, modified $D_\nu$ - Morrey space, Dunkl operator.

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## 1. Introduction

Morrey spaces, introduced by Morrey [22], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. However, in the theory of partial differential equations, along with Morrey spaces, modified Morrey spaces also play an important role [1, 3, 8, 9, 19, 20, 25, 26]. The modified Morrey spaces $\widetilde{L}_{p,\lambda}(R, dm_\nu)$, introduced by V.P. Il'in [14]. The modified Morrey spaces $L_{p,\lambda}(R^n)$ are the set of all locally integrable functions $f$ with the finite (quasi-)norm

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at $x \in R^n$ with radius $t > 0$, $[t]_1 = \min\{1, t\}$, see [9, 13].

On the real line, the Dunkl operators $\Lambda_\nu$ are differential-difference operators introduced in 1989 by Dunkl [4]. For a real parameter $\nu \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $Z_2$ on $R$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \quad x \in R.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and $m_\nu$ be the weighted Lebesgue measure on $R$ given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1}|x|^{2\nu+1}dx, \quad x \in R.$$

For any $x \in R$ and $r > 0$, let
$B(x,r) := \{y \in R : |y| \in \left]\max\{0,|x|-r\},|x|+r\right[\}$ be a Dunkl-ball in $R$.
Then $B(0,r) = \left]-r,r\right[$ and $m_\nu B(0,r) = c_\nu r^{2\nu+2}$, where
$c_\nu := \left[2^{\nu+1}(\nu+1)\Gamma(\nu+1)\right]^{-1}$.

The maximal operator $M_\nu$ associated by Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0}\left(m_\nu(B(x,r))\right)^{-1}\int_{B(x,r)}|f(y)|dm_\nu(y), \quad x \in R$$

and fractional maximal operator $M_{\alpha,\nu}, 0 \le \alpha < 2\nu+2$ associated by Dunkl operator on the real line is given by

$$M_{\alpha,\nu} f(x) := \sup_{r>0}\left(m_\nu(B(x,r))\right)^{-1+\frac{\alpha}{2\nu+2}}\int_{B(x,r)}|f(y)|dm_\nu(y), \quad x \in R.$$

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [29]). Also the fractional maximal function and the fractional integral, associated with $D_\nu$ differential-difference Dunkl operators play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [10, 11, 19, 21]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [5]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [2], V.S. Guliyev and Y.Y. Mammadov [10, 11, 12], Y.Y. Mammadov [16, 18], L. Kamoun [15], M.A. Mourou [23], F. Soltani [27, 28], K. Trimeche [30] and others. Moreover, the results on $L_\Phi(R, dm_\nu)$-boundedness of fractional maximal operator and its commutators associated with $D_\nu$ were obtained in [12, 17].

It is well known that maximal operator play an important role in harmonic analysis (see [29]). Harmonic analysis associated to the Dunkl transform and the

Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $R$, we study the boundedness of the fractional maximal operator $M_{\alpha,\nu}$, on modified $D_\nu$ -Morrey spaces $\widetilde{L}_{p,\lambda}(R, dm_\nu)$.

By A≲B we mean that A≤CB with some positive constant C independent of appropriate quantities. If A≲B and B≲A, we write A≈B and say that A and B are equivalent.

## 2. Preliminaries in the Dunkl setting on $R$

**Definition 1.** Let $0 < p < \infty, \lambda \in R, \mu \in R, [t]_1 = \min\{1, t\}, t > 0$. We denote by $L_{p,\lambda}(R, dm_\nu)$ the Morrey space [24] (≡ $D_\nu$ -Morrey space), and by $\widetilde{L}_{p,\lambda}(R, dm_\nu)$ the modified Morrey space [24] (≡ modified $D_\nu$ -Morrey space), associated with the Dunkl operator the set of all classes of locally integrable functions $f$ with the finite norms

$$\|f\|_{L_{p,\lambda}(R,dm_\nu)} = \sup_{x\in R^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)} = \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

*respectively.*

**Definition 2.** Let $0 < p < \infty, \lambda \in R$ and $\mu \in R$. We denote the weak Morrey space $L_{p,\lambda}(R, dm_\nu)$ [24] (≡ weak $D_\nu$ -Morrey space), and the weak modified Morrey space $\widetilde{L}_{p,\lambda}(R, dm_\nu)$ [24] (≡ weak modified $D_\nu$ -Morrey space), associated with the Dunkl operator the set of all classes of locally integrable functions $f$ with the finite norms

$$\|f\|_{WL_{p,\lambda}(R,dm_\nu)} = \sup_{x\in R, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(R,dm_\nu)} = \sup_{x\in R, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

respectively.

**Lemma 1.** [24] *If* $0 < p < \infty, 0 \le \lambda \le 2\nu + 2$, then

$$\widetilde{L}_{p,\lambda}(R,dm_\nu)=L_{p,\lambda}(R,dm_\nu)\cap L_p(R,dm_\nu)$$

*and*

$$\|f\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)}=\max\left\{\|f\|_{L_{p,\lambda}(R,dm_\nu)},\|f\|_{L_p(R,dm_\nu)}\right\}.$$

**Lemma 2.** [24] *If* $0<p<\infty$, $0\le\lambda\le 2\nu+2$ then

$$W\widetilde{L}_{p,\lambda,\mu}(R,dm_\nu)=WL_{p,\lambda,\mu}(R,dm_\nu)\cap WL_p(R,dm_\nu)$$

*and*

$$\|f\|_{W\widetilde{L}_{p,\lambda,\mu}(R,dm_\nu)}=\max\left\{\|f\|_{WL_{p,\lambda,\mu}(R,dm_\nu)},\|f\|_{WL_p(R,dm_\nu)}\right\}.$$

**Remark 1.** [24] *If* $0<p<\infty$*, and* $\lambda>2\nu+2$, then

$$\widetilde{L}_{p,\lambda}(R,dm_\nu)=W\widetilde{L}_{p,\lambda}(R,dm_\nu)=\Theta(R),$$

where $\Theta\equiv\Theta(R)$ is the set of all functions equivalent to $0$ on $R$.

### 3. Fractional maximal operator $M_{\alpha,\nu}$ in modified Morrey spaces $\widetilde{L}_{p,\lambda}(R,dm_\nu)$

In this section, we investigate the boundedness of the fractional maximal operator Mα,ν in modified Morrey spaces $\widetilde{L}_{p,\lambda}(R,dm_\nu)$.

The following Guliyev type local estimates are valid (see also [6, 7]).

**Lemma 3.** Let $0\le\alpha<2\nu+2, 1\le p<\dfrac{2\nu+2}{\alpha}, \dfrac{1}{p}-\dfrac{1}{q}=\dfrac{\alpha}{2\nu+2}$ and $B(x,r)$ be any Dunkl-ball in $R$. If $p>1$, then the inequality

$$\|M_{\alpha,\nu}f\|_{L_q(B(x,r),dm_\nu)}\lesssim\|f\|_{L_p(2B,dm_\nu)}+r^{\frac{2\nu+2}{q}}\sup_{t>2r}t^{-2\nu-2+\alpha}\|f\|_{L_1(B(x,t),dm_\nu)} \tag{1}$$

holds for all $f\in L_p^{loc}(R,dm_\nu)$.

Moreover if $p=1$, then the inequality

$$\|M_{\alpha,\nu}f\|_{WL_q(B(x,r),dm_\nu)}\lesssim\|f\|_{L_1(2B,dm_\nu)}+r^{\frac{2\nu+2}{q}}\sup_{t>2r}t^{-2\nu-2+\alpha}\|f\|_{L_1(B(x,t),dm_\nu)} \tag{2}$$

holds for all $f \in L_1^{loc}(R, dm_\nu)$.

Proof. Let $0 \le \alpha < 2\nu + 2, 1 \le p < \frac{2\nu+2}{\alpha}, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$. For arbitrary Dunkl-ball
$B = B(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{{}^C(2B)}$

$$\|M_{\alpha,\nu} f\|_{L_q(B,dm_\nu)} \lesssim \|M_{\alpha,\nu} f_1\|_{L_q(B,dm_\nu)} + \|M_{\alpha,\nu} f_2\|_{L_q(B,dm_\nu)}.$$

By the continuity of the operator $M_{\alpha,\nu} : L_p(R, dm_\nu) \to L_q(R, dm_\nu)$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$ see, for example, [24]) we have

$$\|M_{\alpha,\nu} f_1\|_{L_q(B,dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)}.$$

Let $y$ be an arbitrary point from $B$. If $B(y,\tau) \cap {}^C(2B) \ne \emptyset$, then $\tau > r$. Indeed, if $z \in B(y,\tau) \cap {}^C(2B)$, then $\tau > |y - z| \ge |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y,\tau) \cap {}^C(2B) \subset B(x, 2\tau)$. Indeed, $z \in B(y,\tau) \cap {}^C(2B)$, then we get $|x - z| \le |y - z| + |x - y| < \tau + r < 2\tau$.

Hence

$$M_{\alpha,\nu} f_2(y) = \sup_{\tau>0} \frac{1}{m_\nu(B(y,\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(y,\tau)\cap{}^C(2B)} |f(z)| dm_\nu(z)$$

$$\le 2^{2\nu+2-\alpha} \sup_{\tau>r} \frac{1}{m_\nu(B(x,2\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(x,2\tau)} |f(z)| dm_\nu(z)$$

$$= 2^{2\nu+2-\alpha} \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)| dm_\nu(z).$$

Therefore, for all $y \in B$ we have

$$M_{\alpha,\nu} f_2(y) \le 2^{2\nu+2-\alpha} \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)| dm_\nu(z). \tag{3}$$

Applying Hölder's inequality, we get

$$M_{\alpha,\nu} f_2(y) \lesssim \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|^p dm_\nu(z) \tag{4}$$

Thus

$$\|M_{\alpha,\nu} f\|_{L_q(B,dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)} + m_\nu(B(x,\tau))^{\frac{1}{q}} \left( \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)| dm_\nu(z) \right)$$

.

Let $p=1$. It is obvious that for any ball $B=B(x,r)$

$$\|M_{\alpha,\nu} f\|_{WL_q(B,dm_\nu)} \le \|M_{\alpha,\nu} f_1\|_{WL_q(B,dm_\nu)} + \|M_{\alpha,\nu} f_2\|_{WL_q(B,dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_1(R,dm_\nu) \to WL_q(R,dm_\nu)$ we have

$$\|M_{\alpha,\nu} f_1\|_{WL_q(B,dm_\nu)} \lesssim \|f\|_{L_1(2B,dm_\nu)}.$$

Then by (4) we get the inequality (2).

**Lemma 4.** Let $0\le\alpha<2\nu+2, 1\le p<\frac{2\nu+2}{\alpha}, \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\nu+2}$ and $B(x,r)$ be any Dunkl-ball in $R$. If $p>1$, then the inequality

$$\|M_{\alpha,\nu} f\|_{L_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,t),dm_\nu)} \tag{5}$$

holds for all $f\in L_p^{loc}(R,dm_\nu)$.

Moreover if $p=1$, then the inequality

$$\|M_{\alpha,\nu} f\|_{WL_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_1(B(x,t),dm_\nu)} \tag{6}$$

holds for all $f\in L_1^{loc}(R,dm_\nu)$.

Proof. Let $0\le\alpha<2\nu+2, 1\le p<\frac{2\nu+2}{\alpha}, \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\nu+2}$. Denote

$$A_1 = m_\nu(B(x,\tau))^{\frac{1}{q}} \left( \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|\, dm_\nu(z) \right),$$

$$A_2 = \|f\|_{L_p(2B,dm_\nu)}\ .$$

Applying Hölder's inequality, we get

$$A_1 \lesssim m_\nu(B(x,\tau))^{\frac{1}{q}} \left( \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \int_{B(x,\tau)} |f(z)|^p\ dm_\nu(z) \right)^{\frac{1}{p}}.$$

On the other hand,

$$m_\nu(B(x,\tau))^{\frac{1}{q}} \left( \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \int_{B(x,\tau)} |f(z)|^p\ dm_\nu(z) \right)^{\frac{1}{p}}$$

$$\gtrsim m_\nu(B(x,\tau))^{\frac{1}{q}} \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \|f\|_{L_p(2B,dm_\nu)}.$$

Since by Lemma 3.

$$\left\|M_{\alpha,\nu} f\right\|_{L_q(B,dm_\nu)} \lesssim A_1 + A_2,$$

we arrive at (5).

Let $p=1$. Inequality (6) directly follows from (2).

The following Spanne's type result completely characterizes the boundedness of $M_{\alpha,\nu}$ on modified Morrey spaces $\widetilde{L}_{p,\lambda}(R,dm_\nu)$.

**Theorem 1.** Let $0 \le \alpha < 2\nu+2, 0 \le \lambda < 2\nu+2, 1 \le p < \frac{2\nu+2-\lambda}{\alpha}$, and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\nu+2}$

1. If $f \in \widetilde{L}_{1,\lambda}(R,dm_\nu)$, then $M_{\alpha,\nu} f \in W\widetilde{L}_{q,\lambda q}(R,dm_\nu)$ and

$$\left\|M_{\alpha,\nu} f\right\|_{L_{q,\lambda q}(R,dm_\nu)} \le C_{1,\lambda} \left\|f\right\|_{L_p(R,dm_\nu)}, \tag{7}$$

where $C_{q,\lambda}$ is independent of $f$.

1. If $f \in \widetilde{L}_{p,\lambda}(R,dm_\nu)$, $1<p<\infty$ then $M_\nu f \in \widetilde{L}_{q,\frac{\lambda q}{p}}(R,dm_\nu)$ and

$$\left\|M_{\alpha,\nu} f\right\|_{\widetilde{L}_{q,\frac{\lambda q}{p}}(R,dm_\nu)} \le C_{p,q,\lambda} \left\|f\right\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)}, \tag{8}$$

where $C_{p,\lambda}$ is independent of $p, \lambda$ and $\nu$.

Proof. Let $p=1$, $f \in \widetilde{L}_{1,\lambda}(R,dm_\nu)$. From the inequality (2) we get

$$\left\|M_{\alpha,\nu} f\right\|_{W\widetilde{L}_{q,\lambda q}(R,dm_\nu)} = \sup_{x\in R^n, t>0} [t]_1^{-\lambda} \left\|M_{\alpha,\nu} f\right\|_{WL_q(B(x,t),dm_\nu)}$$

$$\lesssim \sup_{x\in R^n, t>0} [t]_1^{-\lambda} t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \left\|f\right\|_{L_1(B(x,\tau))}$$

$$\lesssim \left\|f\right\|_{\widetilde{L}_{1,\lambda,\mu}(R,dm_\nu)} \sup_{x\in R^n, t>0} [t]_1^{-\lambda} t^{\frac{2\nu+2}{q}} \sup_{\tau>t} \tau^{-\frac{2\nu+2}{q}} [\tau]_1^{\lambda}$$

$$= \left\|f\right\|_{\widetilde{L}_{1,\lambda,\mu}(R,dm_\nu)} \sup_{x\in R^n, t>0} [t]_1^{\frac{2\nu+2}{q}-\lambda} \sup_{\tau>t} [\tau]_1^{\lambda-\frac{2\nu+2}{q}} \lesssim \left\|f\right\|_{\widetilde{L}_{1,\lambda}(R,dm_\nu)}$$

which implies that the operator $M_{\alpha,\nu}$ is bounded from $\widetilde{L}_{1,\lambda}(R,dm_\nu)$ to $W\widetilde{L}_{q,\lambda}(R,dm_\nu)$.

Let $1<p<\infty$, $f \in \widetilde{L}_{p,\lambda}(R,dm_\nu)$. From the inequality (1) we get

$$\left\|M_{\alpha,\nu} f\right\|_{\widetilde{L}_{q,\frac{\lambda q}{p}}(R,dm_\nu)} = \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \left\|M_{\alpha,\nu} f\right\|_{L_p(B(x,t),dm_\nu)}$$

$$\lesssim \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,\tau))}$$

$$\lesssim \|f\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)} \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{-\alpha+\frac{2\nu+2}{p}} \sup_{\tau>t} \tau^{\alpha-\frac{2\nu+2}{p}} [\tau]_1^{\frac{\lambda}{p}}$$

$$= \|f\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)} \sup_{x\in R^n, t>0} [t]_1^{-\alpha+\frac{2\nu+2-\lambda}{p}} \sup_{\tau>t} [\tau]_1^{\alpha-\frac{2\nu+2-\lambda}{p}} \lesssim \|f\|_{\widetilde{L}_{p,\lambda}(R,dm_\nu)}$$

which implies that the operator $M_{\alpha,\nu}$ is bounded from $\widetilde{L}_{p,\lambda}(R,dm_\nu)$. to $W\widetilde{L}_{q,\lambda}(R,dm_\nu)$.

## 4. Conclusion

We find sufficient conditions for the boundedness of the operator $M_{\alpha,\nu}$ on modified $D_\nu$ - Morrey spaces $\widetilde{L}_{p,\lambda}(R,dm_\nu)$.

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